Regular Languages and Finite State Automata

Data structures and algorithms for Computational Linguistics III

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Why study finite-state automata?

- Unlike some of the abstract machines we discussed, finite-state automata are efficient models of computation
- There are many applications
  - Electronic circuit design
  - Workflow management
  - Games
  - Pattern matching
  - …

  But More importantly ;)
  - Tokenization, stemming
  - Morphological analysis
  - Shallow parsing/chunking
  - …

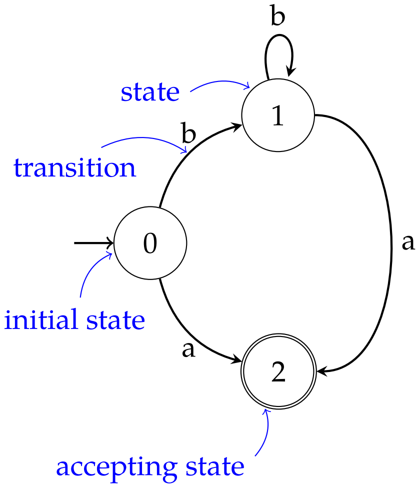
Finite-state automata (FSA)

- A finite-state machine is in one of a finite-number of states in a given time
- The machine changes its state based on its input
- Every regular language is generated/recognized by an FSA
- Every FSA generates/recognizes a regular language
- Two flavors:
  - *Deterministic finite automata* (DFA)
  - *Non-deterministic finite automata* (NFA)

  Note: the NFA is a superset of DFA.

DFA as a graph

- States are represented as nodes
- Transitions are shown by the edges, labeled with symbols from an alphabet
- One of the states is marked as the *initial state*
- Some states are accepting states



DFA: formal definition

Formally, a finite state automaton, M, is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

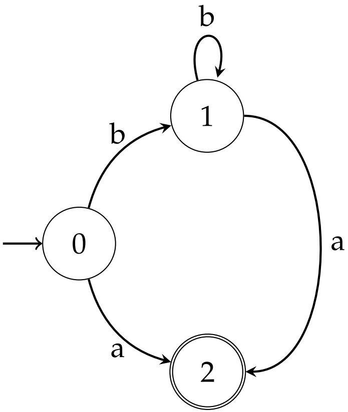
- Σ is the alphabet, a finite set of symbols
- Q a finite set of states
- q_0 is the start state, $q_0 \in Q$
- F is the set of final states, $F \subseteq Q$
- Δ is a function that takes a state and a symbol in the alphabet, and returns another state ($\Delta : Q \times \Sigma \rightarrow Q$)

At any given time, for any input,
a DFA has a single well-defined action to take.

DFA: formal definition

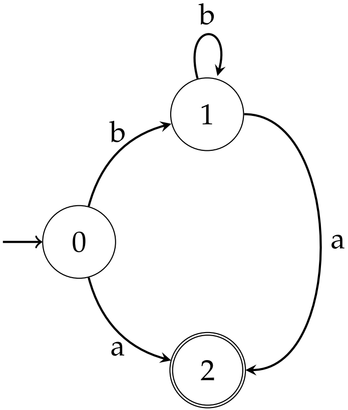
an example

$$
\begin{aligned}
\Sigma &= \{a, b\} \\
Q &= \{q_0, q_1, q_2\} \\
q_0 &= q_0 \\
F &= \{q_2\} \\
\Delta &= \{(q_0, a) \rightarrow q_2, \\
&\qquad (q_0, b) \rightarrow q_1, \\
&\qquad (q_1, a) \rightarrow q_2, \\
&\qquad (q_1, b) \rightarrow q_1\}
\end{aligned}
$$



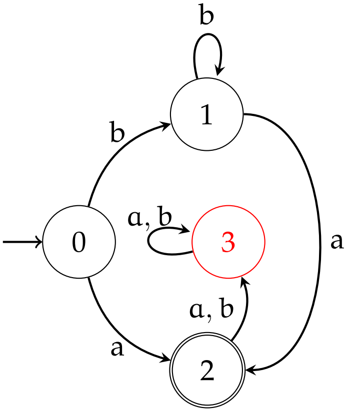
Another note on DFA

- Is this FSA deterministic?



Another note on DFA

- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state



Another note on DFA

- Is this FSA deterministic?
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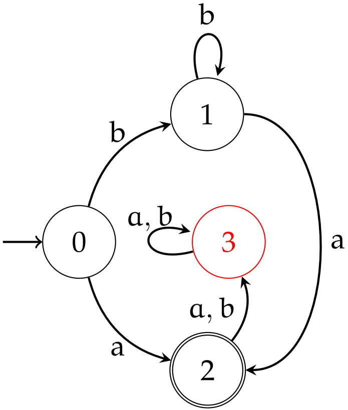
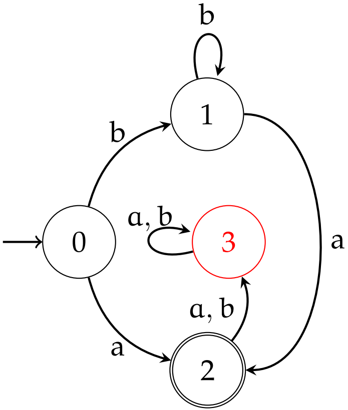
- For brevity, we skip the explicit error state



Another note on DFA

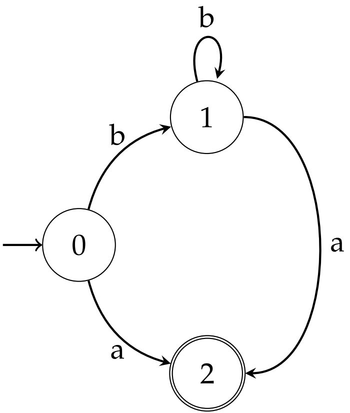
- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state
- For brevity, we skip the explicit error state
  - In that case, when we reach a dead end, recognition fails



DFA: the transition table

transition table

		symbol	
		a	**b**
state	$\rightarrow$**0**	2	1
	1	2	1
	***2**	$\varnothing$	$\varnothing$

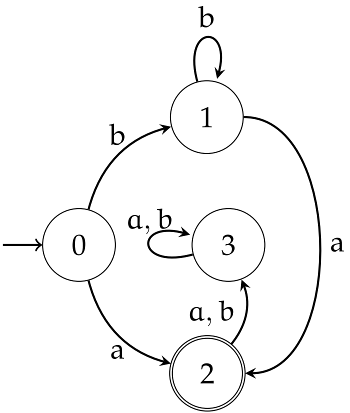


- $\rightarrow$ marks the start state
- * marks the accepting state(s)

DFA: the transition table

transition table

		symbol	
		a	**b**
state	→**0**	2	1
	1	2	1
	***2**	3	3
	3	3	3

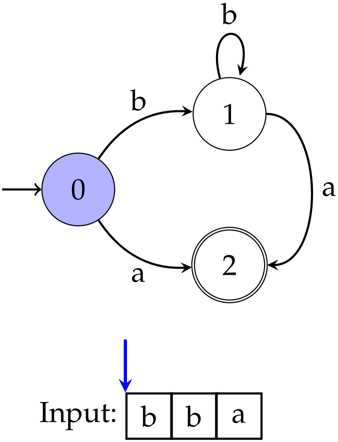


→ marks the start state

* marks the accepting state(s)

DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input



DFA recognition

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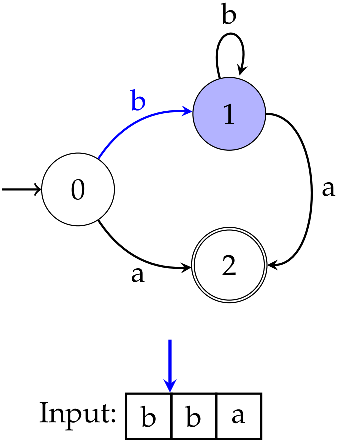

DFA recognition

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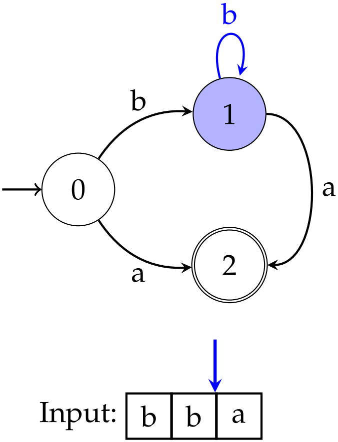

DFA recognition

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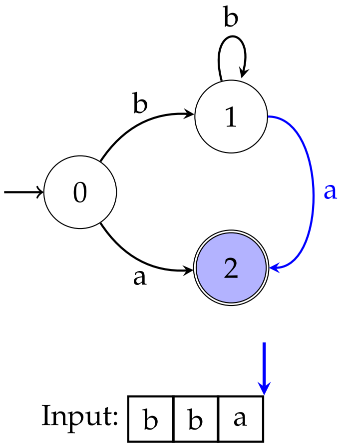
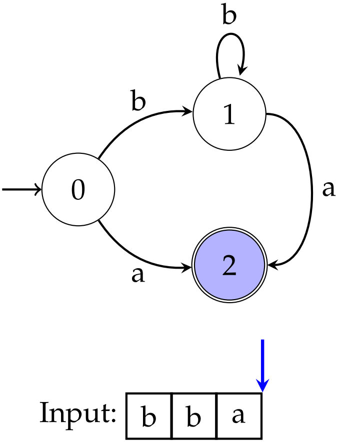


DFA recognition

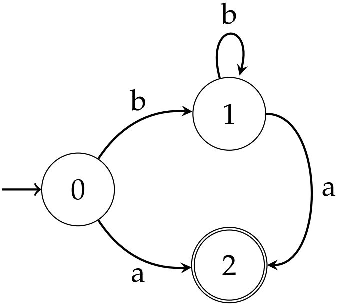
1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input



DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input

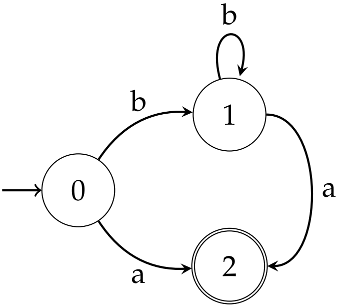
- What is the complexity of the algorithm?
- How about inputs:
  - bbbb
  - aa



Input: | b | b | a |

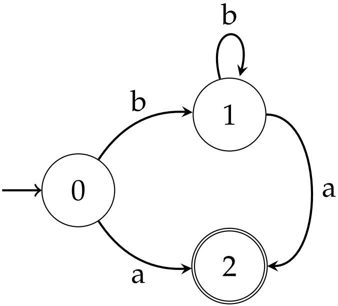
A few questions

- What is the language recognized by this FSA?



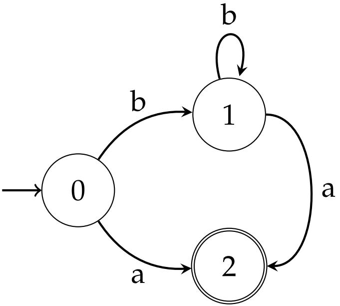
A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?



A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?
- Draw a DFA recognizing strings with even number of ‘a’s over $\Sigma = \{a, b\}$



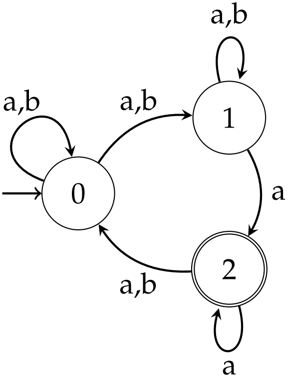
Non-deterministic finite automata

Formal definition

A non-deterministic finite state automaton, M, is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

- Σ is the alphabet, a finite set of symbols
- Q a finite set of states
- q_0 is the start state, $q_0 \in Q$
- F is the set of final states, $F \subseteq Q$
- Δ is a function from (Q, Σ) to $P(Q)$, power set of Q ($\Delta : Q \times \Sigma \rightarrow P(Q)$)

An example NFA



transition table

		symbol	
		a	**b**
state	→**0**	0,1	0,1
	1	1,2	1
	***2**	0,2	0

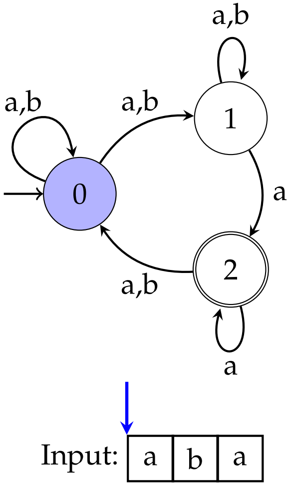
- We have nondeterminism, e.g., if the first input is a, we need to choose between states 0 or 1
- Transition table cells have *sets* of states

Dealing with non-determinism

- Follow one of the links, store alternatives, and *backtrack* on failure
- Follow all options in parallel
- Use dynamic programming (e.g., as in chart parsing)

NFA recognition

as search (with backtracking)



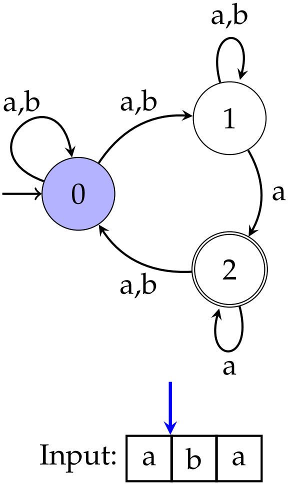
Agenda

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



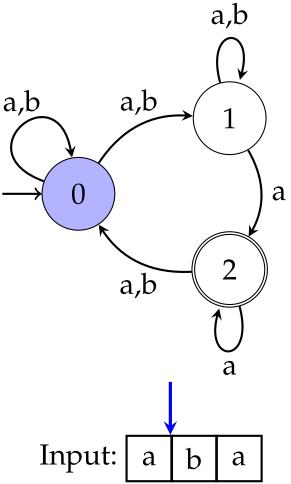
Agenda
$(q_0, 1)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept: if in an accepting state
Reject: not in accepting state & agenda empty
Backtrack: otherwise

NFA recognition

as search (with backtracking)



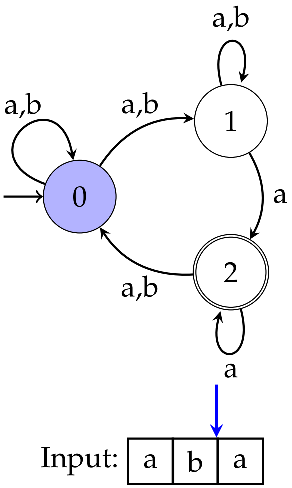
Agenda
$(q_0, 1)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept if in an accepting state
Reject not in accepting state & agenda empty
Backtrack otherwise

NFA recognition

as search (with backtracking)



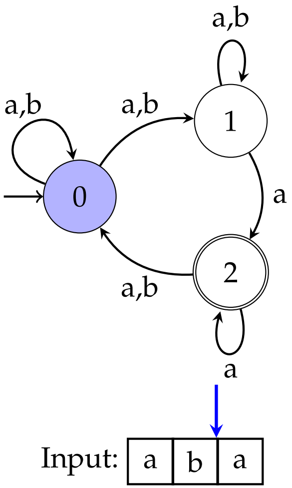
Agenda
$(q_0, 2)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



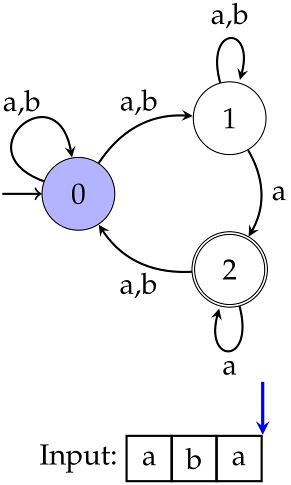
Agenda
$(q_0, 2)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept if in an accepting state
Reject not in accepting state & agenda empty
Backtrack otherwise

NFA recognition

as search (with backtracking)



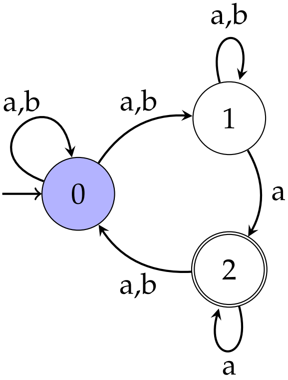
Agenda
$(q_0, 3)$
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept: if in an accepting state
Reject: not in accepting state & agenda empty
Backtrack: otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 3)$
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

Input:

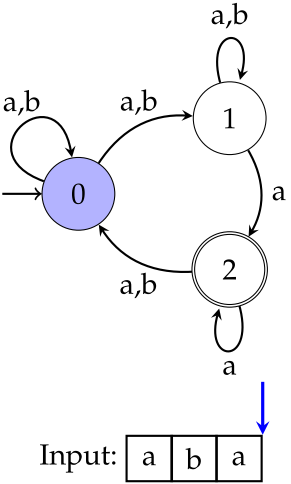
a	b	a

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept: if in an accepting state
Reject: not in accepting state & agenda empty
Backtrack: otherwise

NFA recognition

as search (with backtracking)



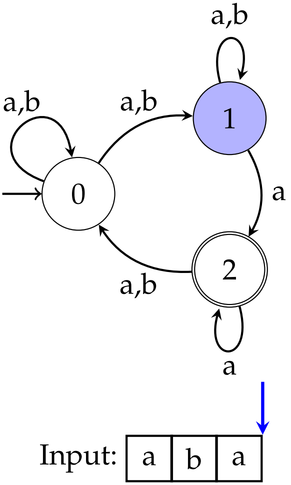
Agenda
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept — if in an accepting state
Reject — not in accepting state & agenda empty
Backtrack — otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept if in an accepting state
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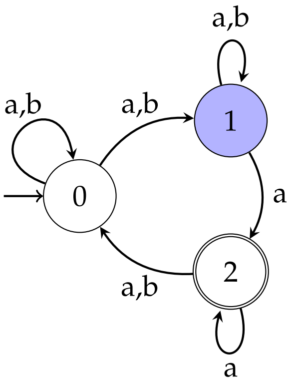
Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

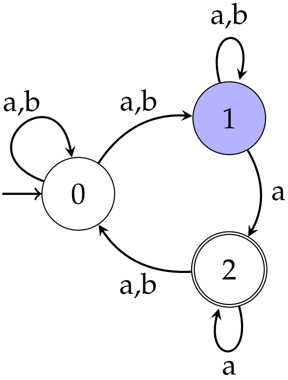
Accept if in an accepting state
Reject not in accepting state & agenda empty
Backtrack otherwise

Input:

a	b	a

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 2)$
$(q_1, 1)$

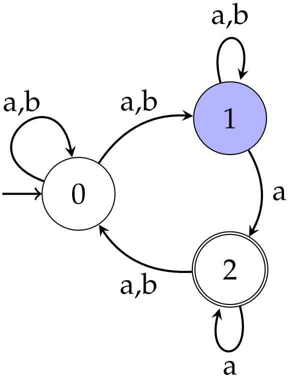
Input: | a | b | a |

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept: if in an accepting state
Reject: not in accepting state & agenda empty
Backtrack: otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_2, 3)$
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept if in an accepting state

Reject not in accepting state & agenda empty

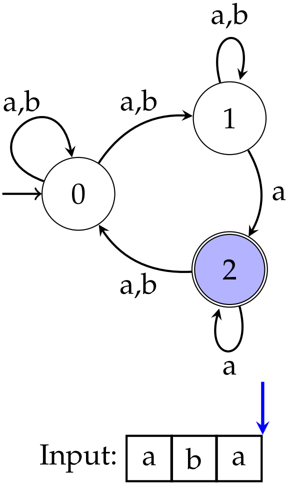
Backtrack otherwise

Input:

a	b	a

NFA recognition

as search (with backtracking)



Agenda
$(q_2, 3)$
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

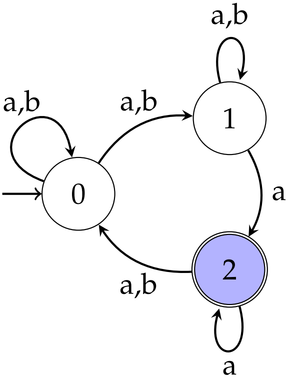
Accept if in an accepting state

Reject not in accepting state & agenda empty

Backtrack otherwise

NFA recognition

as search (with backtracking)



Input:

a	b	a

Agenda
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

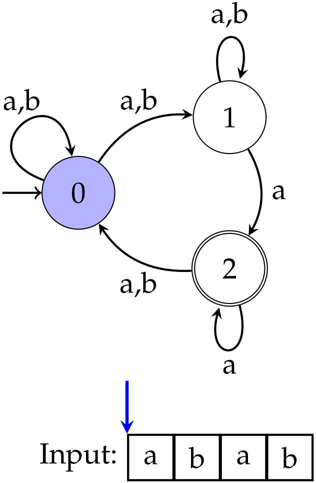
NFA recognition as search

summary

- Worst time complexity is exponential
  - Complexity is worse if we want to enumerate all derivations
- We used a stack as *agenda*, performing a depth-first search
- A queue would result in breadth-first search
- If we have a reasonable heuristic A* search may be an option
- Machine learning methods may also guide finding a fast or the best solution

NFA recognition

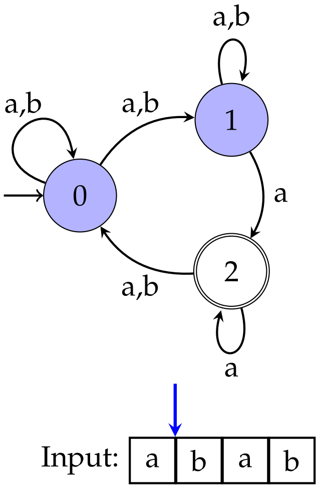
parallel version



1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

NFA recognition

parallel version



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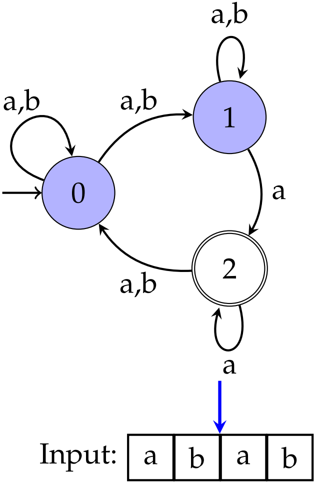
NFA recognition

parallel version



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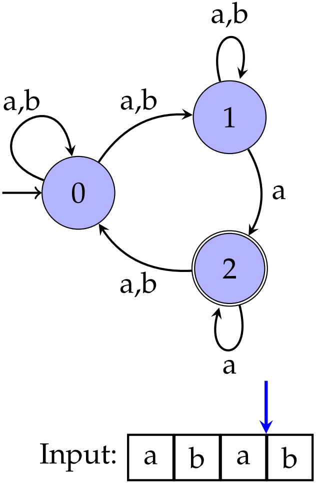
NFA recognition

parallel version



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NFA recognition

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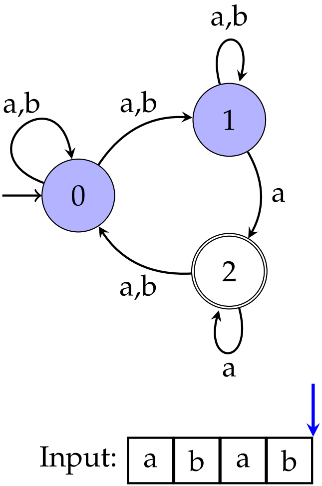

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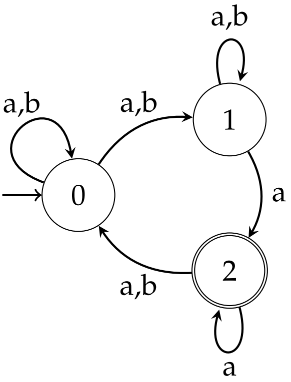
NFA recognition

parallel version



Input:

a	b	a	b

1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

Note: the process is *deterministic*, and *finite-state*.

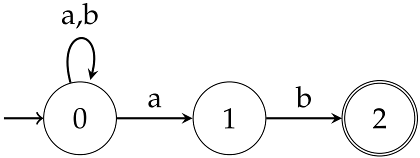
An exercise

Construct an NFA and a DFA for the language over $\Sigma = \{a, b\}$ where all string end with ab.

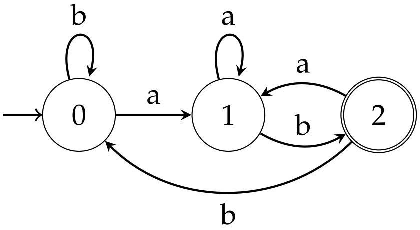
An exercise

Construct an NFA and a DFA for the language over $\Sigma = \{a, b\}$ where all string end with ab.

NFA:

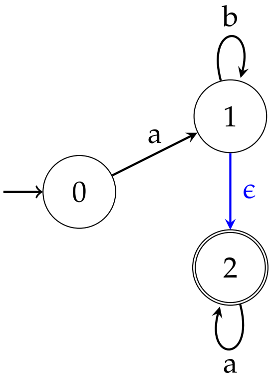


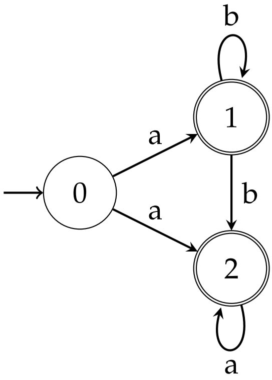
DFA:



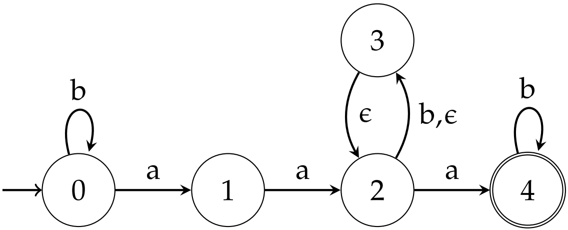
One more complication: ϵ transitions

- An extension of NFA, ϵ-NFA, allows moving without consuming an input symbol, indicated by an ϵ-transition
- Any ϵ-NFA can be converted to an NFA





ϵ-transitions need attention



- How does the (depth-first) NFA recognition algorithm we described earlier work on this automaton?

NFA–DFA equivalence

- The language recognized by every NFA is recognized by some DFA
- The set of DFA is a subset of the set of NFA (a DFA is also an NFA)
- The same is true for ϵ-NFA
- All recognize/generate regular languages
- NFA can automatically be converted to the equivalent DFA

Why do we use an NFA then?

- NFA (or ϵ-NFA) are often easier to construct
  - Intuitive for humans
  - Some representations are easy to convert to NFA rather than DFA, e.g., regular expressions
- NFA may require less memory (fewer states)

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1. Construct (draw) an NFA for the language over $\Sigma = \{a, b\}$, such that 4th symbol from the end is an a

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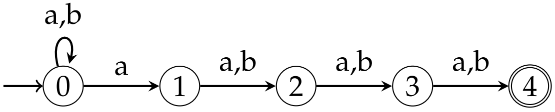

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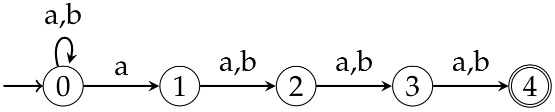
A quick exercise – and a not-so-quick one

1. Construct (draw) an NFA for the language over $\Sigma = \{a, b\}$, such that 4th symbol from the end is an a



2. Construct a DFA for the same language

Determinization

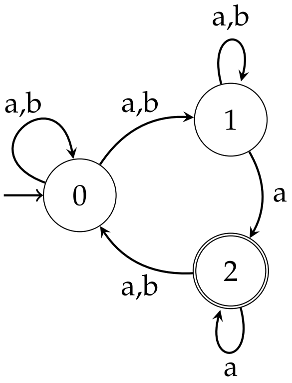
the subset construction

Intuition: remember the parallel NFA recognition. We can consider an NFA being a deterministic machine which is at a set of states at any given time.

- *Subset construction* (sometimes called powerset construction) uses this intuition to convert an NFA to a DFA
- The algorithm can be modified to handle ϵ-transitions (or we can eliminate ϵ's as a pre-processing step)

The subset construction

by example



transition table with subsets

	symbol a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{0\}$	$\{0,1\}$	$\{0,1\}$
$\{1\}$	$\{1,2\}$	$\{1\}$
$* \{2\}$	$\{0,2\}$	$\{0\}$
$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
$* \{0,2\}$	$\{0,1,2\}$	$\{0,1\}$
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The subset construction

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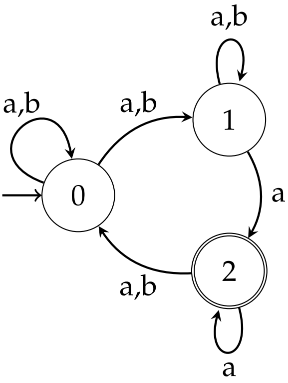

transition table with subsets

	symbol	
	a	**b**
~~$\emptyset$~~	~~$\emptyset$~~	~~$\emptyset$~~
$\rightarrow \{0\}$	$\{0,1\}$	$\{0,1\}$
~~$\{1\}$~~	~~$\{1,2\}$~~	~~$\{1\}$~~
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$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
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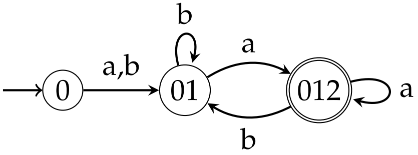
The subset construction

by example: the resulting DFA

transition table without useless/inaccessible states

	symbol a	b
$\rightarrow \{0\}$	$\{0, 1\}$	$\{0, 1\}$
$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1\}$
$* \{0, 1, 2\}$	$\{0, 1, 2\}$	$\{0, 1\}$



The subset construction

by example: the resulting DFA

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	symbol **a**	**b**
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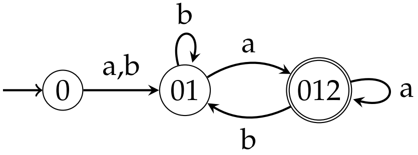
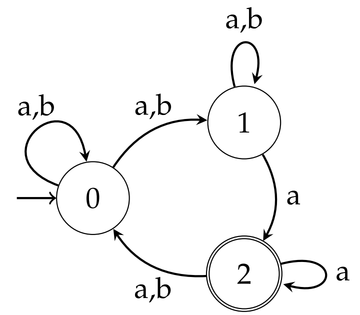


Do you remember the set of states marked during parallel NFA recognition?

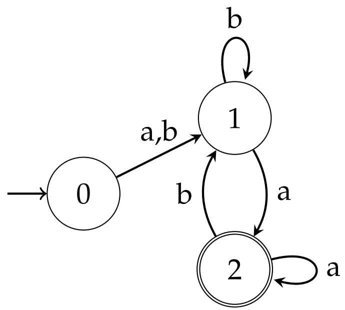
The subset construction

by example: side by side

NFA

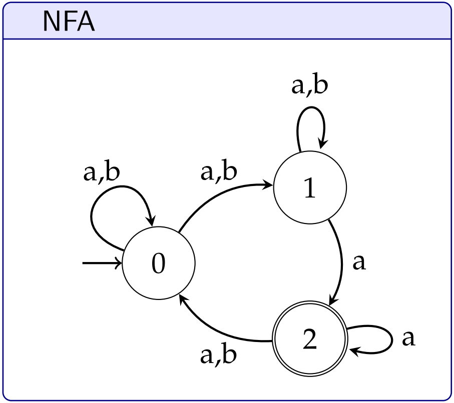


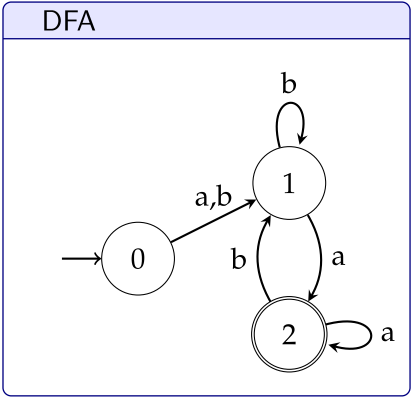
DFA



The subset construction

by example: side by side



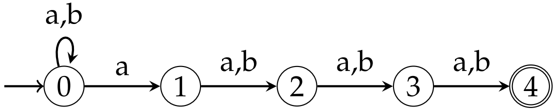


- What language do they recognize?

The subset construction

wrapping up

- In worst case, resulting DFA has 2^n nodes
- Worst case is rather rare, in practice number of nodes in an NFA and the converted DFA are often similar
- In practice, we do not need to enumerate all 2^n subsets
- We've already seen a typical problematic case:

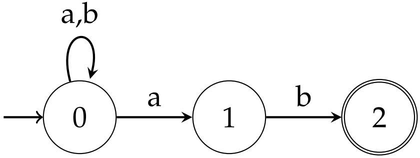


- We can also skip the unreachable states during subset construction

Yet another exercise

Determinize the following automaton

NFA:



Regular languages: definition

A regular grammar is a tuple $G = (\Sigma, N, S, R)$ where

- Σ is an alphabet of terminal symbols
- N are a set of non-terminal symbols
- S is a special 'start' symbol $\in N$
- R is a set of rewrite rules following one of the following patterns ($A, B \in N$, $a \in \Sigma$, ϵ is the empty string)

Left regular

1. $A \rightarrow a$
2. $A \rightarrow Ba$
3. $A \rightarrow \epsilon$

Right regular

1. $A \rightarrow a$
2. $A \rightarrow aB$
3. $A \rightarrow \epsilon$

Regular languages: another definition

A language is regular if there is an FSA that recognizes it

- We denote the language recognized by a finite state automaton M, as $\mathcal{L}(M)$
- The above definition reformulated: if a language L is regular, there is a DFA M, such that $\mathcal{L}(M) = L$
- Remember: any NFA (with or without ϵ transitions) can be converted to a DFA

Some operations on regular languages (and FSA)

L_1L_2 Concatenation of two languages L_1 and L_2: any sentence of L_1 followed by any sentence of L_2

L^* Kleene star of L: L concatenated by itself 0 or more times

L^R Reverse of L: reverse of any string in L

$\overline{L}$ Complement of L: all strings in Σ_L^* except the ones in L $(\Sigma_L^* - L)$

$L_1 \cup L_2$ Union of languages L_1 and L_2: strings that are in any of the languages

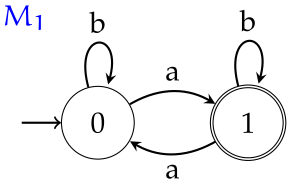
$L_1 \cap L_2$ Intersection of languages L_1 and L_2: strings that are in both languages

Regular languages are closed under all of these operations.

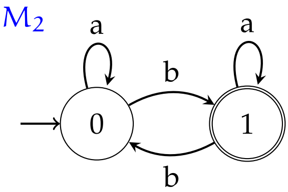
Two example FSA

what languages do they accept?

$L_1 = \mathcal{L}(M_1)$



$L_2 = \mathcal{L}(M_2)$



Two example FSA

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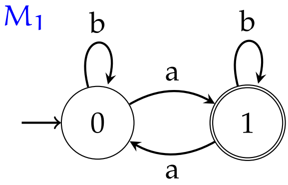
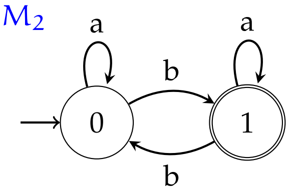


Odd number of a's over $\{a, b\}$.

$L_2 = \mathcal{L}(M_2)$

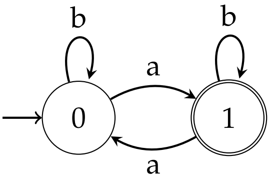


Odd number of b's over $\{a, b\}$.

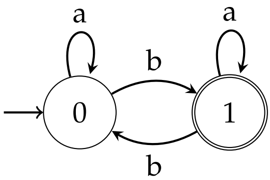
We will use these languages and automata for demonstration.

Concatenation

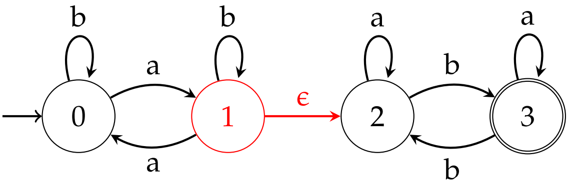
L_1



L_2

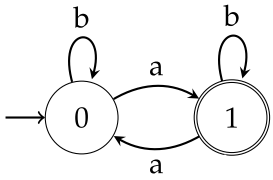


$L_1 L_2$

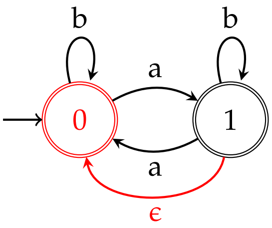


Kleene star

L_1

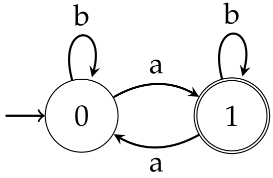


L_1^*

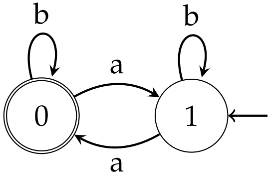


Reversal

L_1

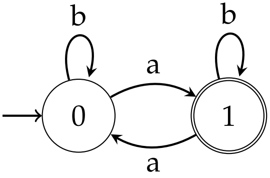


L_1^R

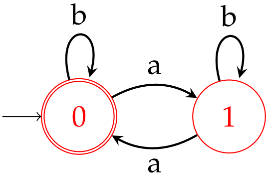


Complement

L_1

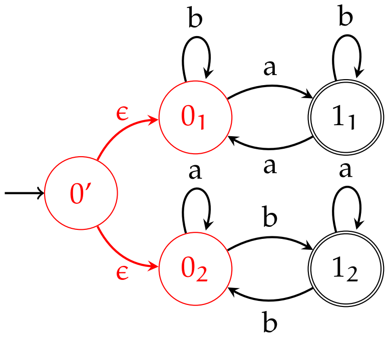


$\overline{L_1}$

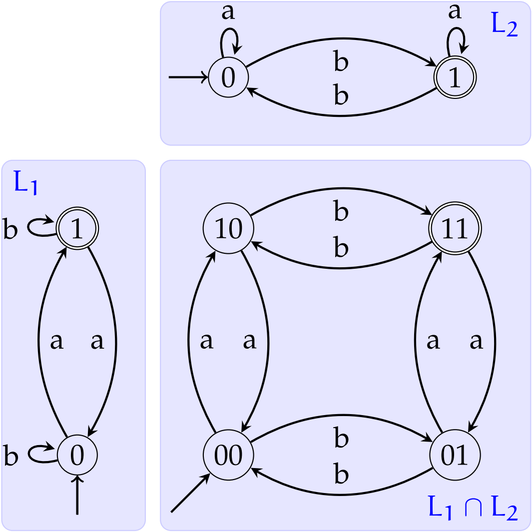


Union

$L_1 \cup L_2$



Intersection



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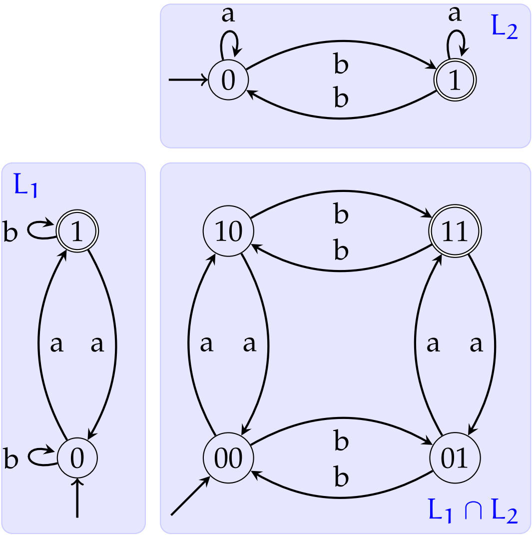


…or

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

Closure properties of regular languages

- Since results of all the operations we studied are FSA: Regular languages are closed under
  - Concatenation
  - Kleene star
  - Reversal
  - Complement
  - Union
  - Intersection

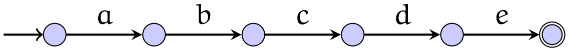
Is a language regular?

— or not

- To show that a language is regular, it is sufficient to find an FSA that recognizes it.
- Showing that a language is *not* regular is more involved
- We will study a method based on *pumping lemma*

Pumping lemma

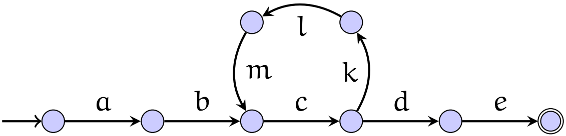
intuition



- What is the length of longest string generated by this FSA?

Pumping lemma

intuition



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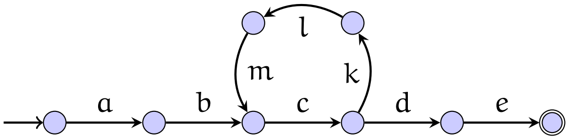
Pumping lemma

intuition



- What is the length of longest string generated by this FSA?
- Any FSA generating an infinite language has to have a loop (application of recursive rule(s) in the grammar)
- Part of every string longer than some number will include repetition of the same substring ('cklm' above)

Pumping lemma

definition

For every regular language L, there exist an integer p such that a string $x \in L$ can be factored as $x = uvw$,

- $uv^iw \in L, \forall i \geqslant 0$
- $v \neq \epsilon$
- $|uv| \leqslant p$

Pumping lemma

definition

For every regular language L, there exist an integer p such that a string $x \in L$ can be factored as $x = uvw$,

- $uv^iw \in L, \forall i \geqslant 0$
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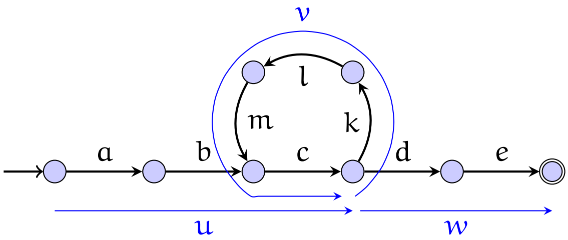


How to use pumping lemma

- We use pumping lemma to prove that a language is not regular
- Proof is by contradiction:
  - Assume the language is regular
  - Find a string x in the language, for all splits of $x = uvw$, at least one of the pumping lemma conditions does not hold
    - $uv^iw \in L$ $(\forall i \geqslant 0)$
    - $v \neq \epsilon$
    - $|uv| \leqslant p$

Pumping lemma example

prove $L = a^n b^n$ is not regular

- Assume L is regular: there must be a p such that, if uvw is in the language
  1. $uv^i w \in L \ (\forall i \geqslant 0)$
  2. $v \neq \epsilon$
  3. $|uv| \leqslant p$
- Pick the string $a^p b^p$
- For the sake of example, assume $p = 5$, $x = aaaaabbbbb$
- Three different ways to split

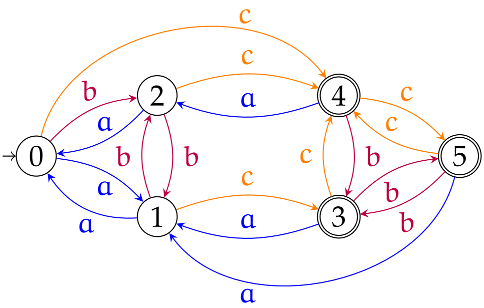
$\underbrace{a}_{u}\underbrace{aaa}_{v}\underbrace{abbbbb}_{w}$	violates 1
$\underbrace{aaaa}_{u}\underbrace{ab}_{v}\underbrace{bbbb}_{w}$	violates 1 & 3
$\underbrace{aaaaab}_{u}\underbrace{bbb}_{v}\underbrace{b}_{w}$	violates 1 & 3

DFA minimization

- For any regular language, there is a unique *minimal* DFA
- By finding the minimal DFA, we can also prove equivalence (or not) of different FSA
- In general the idea is:
  - Throw away unreachable states (easy)
  - Merge equivalent states
- There are two well-known algorithms for minimization:
  - Hopcroft's algorithm: find and eliminate equivalent states by partitioning the set of states
  - Brzozowski's algorithm: 'double reversal'

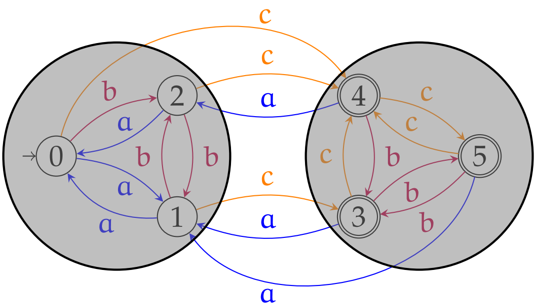
Finding equivalent states

Intuition



Finding equivalent states

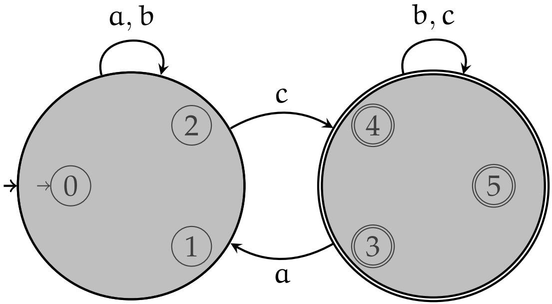
Intuition



The edges leaving the group of nodes are identical.
Their *right languages* are the same.

Finding equivalent states

Intuition



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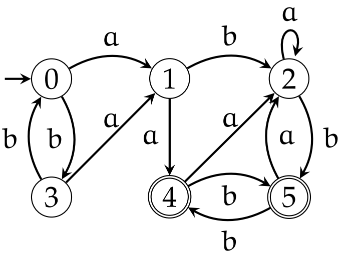
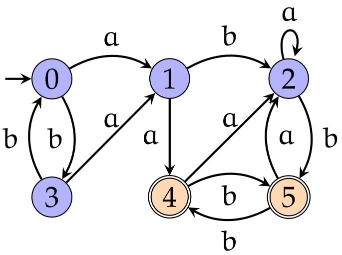
Minimization by partitioning



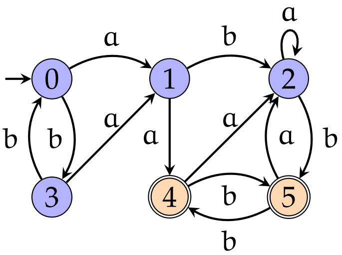
Minimization by partitioning



- Accepting & non-accepting states form a partition

  $Q_2 = \{0, 1, 2, 3\}$, $Q_2 = \{4, 5\}$

Minimization by partitioning



- Accepting & non-accepting states form a partition
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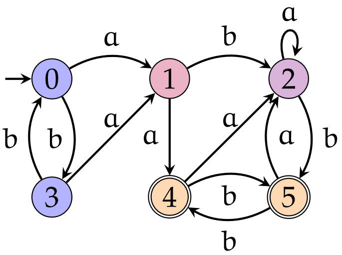
Minimization by partitioning



- Accepting & non-accepting states form a partition
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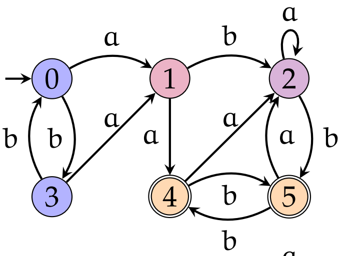
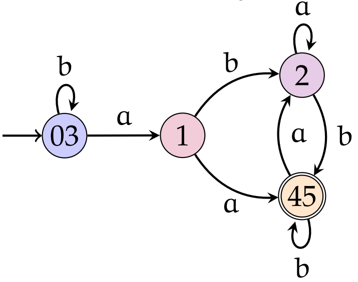
- $Q_1 = \{0, 3\}$, $Q_3 = \{1\}$, $Q_4 = \{2\}$, $Q_2 = \{4, 5\}$

Minimization by partitioning





- Accepting & non-accepting states form a partition
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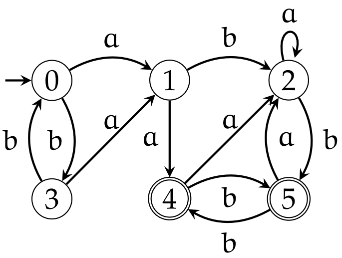
- If any two nodes go to different sets for any of the symbols split
- $Q_1 = \{0, 3\}$, $Q_3 = \{1\}$, $Q_4 = \{2\}$, $Q_2 = \{4, 5\}$
- Stop when we cannot split any of the sets

Minimization by partitioning

tabular version



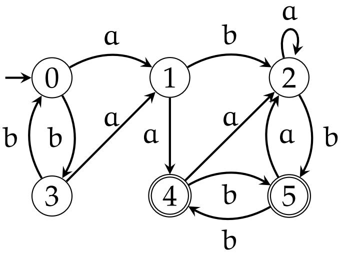
- Create a state-by-state table, mark *distinguishable* pairs: (q_1, q_2) such that $(\Delta(q_1, x), \Delta(q_2, x))$ is a distinguishable pair for any $x \in \Sigma$

1					
2					
3					
4					
5					
	0	1	2	3	4

1

Minimization by partitioning

tabular version



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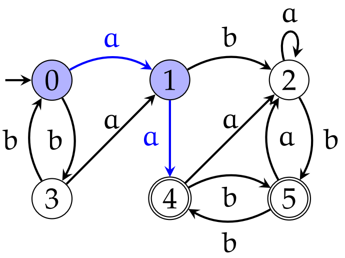
1					
2					
3					
4	●	●	●	●	
5	●	●	●	●	
	0	1	2	3	4

2

Minimization by partitioning

tabular version



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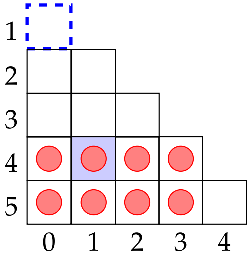
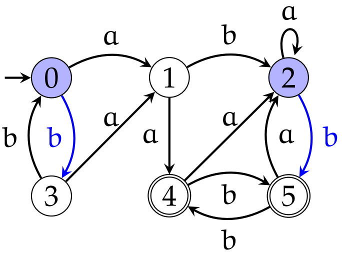


3

Minimization by partitioning

tabular version



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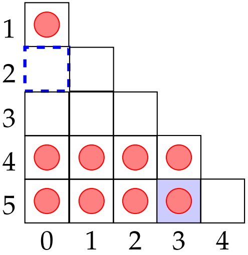
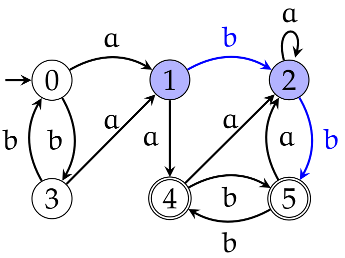


4

Minimization by partitioning

tabular version



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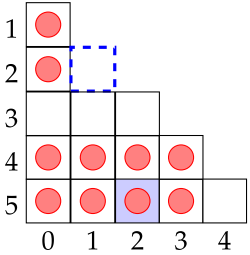
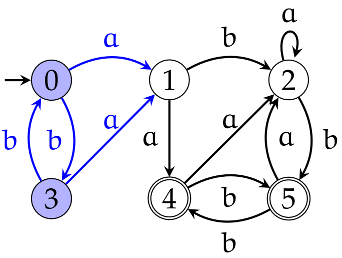


5

Minimization by partitioning

tabular version



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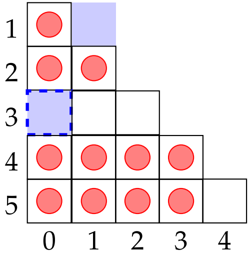
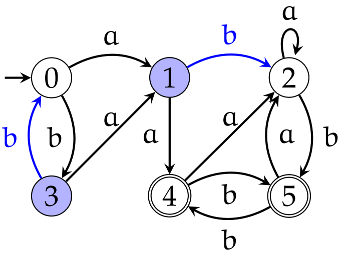


Minimization by partitioning

tabular version



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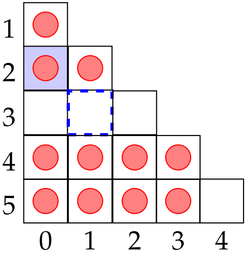
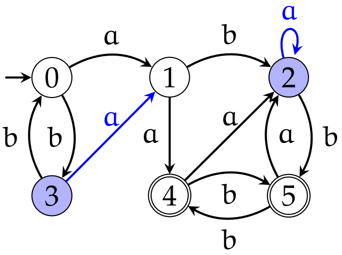


Minimization by partitioning

tabular version



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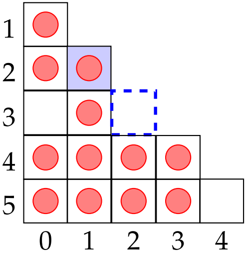
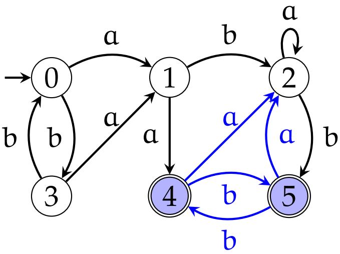


Minimization by partitioning

tabular version



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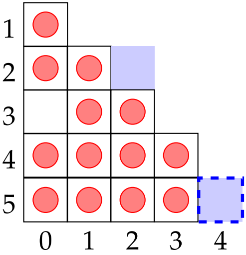
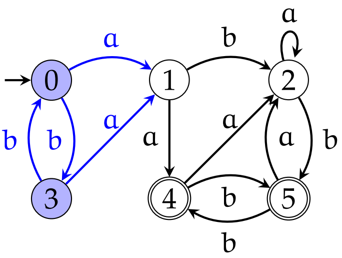


9

Minimization by partitioning

tabular version



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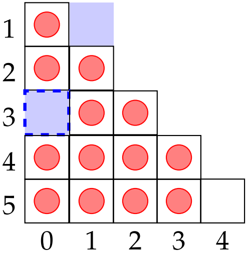
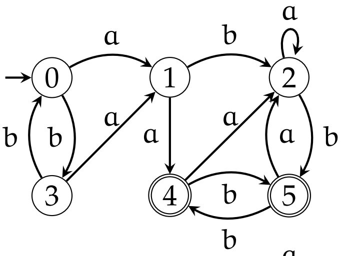


10

Minimization by partitioning

tabular version



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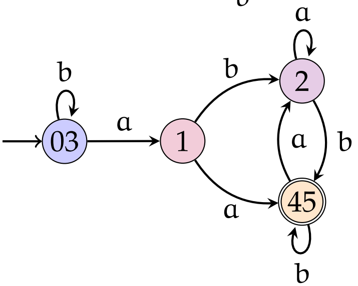
	0	1	2	3	4
1	●				
2	●	●			
3		●	●		
4	●	●	●	●	
5	●	●	●	●	

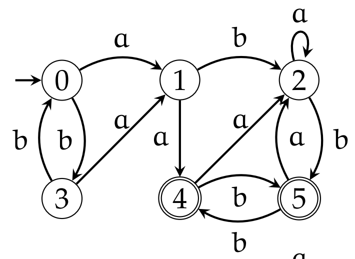


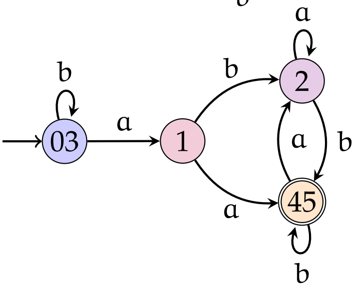
- Merge indistinguishable states

11

Minimization by partitioning

tabular version





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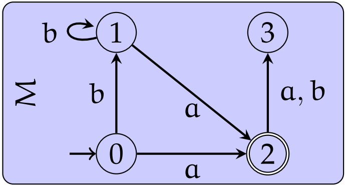
	0	1	2	3	4
1	●				
2	●	●			
3		●	●		
4	●	●	●	●	
5	●	●	●	●	

- Merge indistinguishable states
- The algorithm can be improved by choosing which cell to visit carefully

12

Brzozowski's algorithm

double reverse (r), determinize (d)



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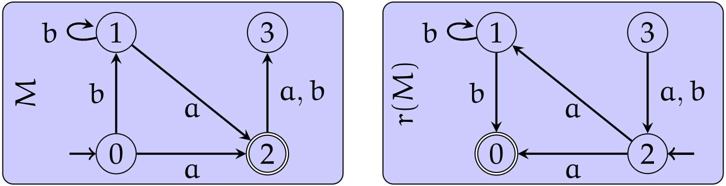

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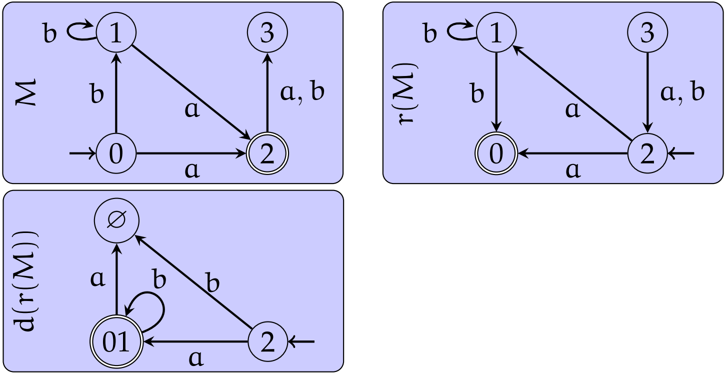

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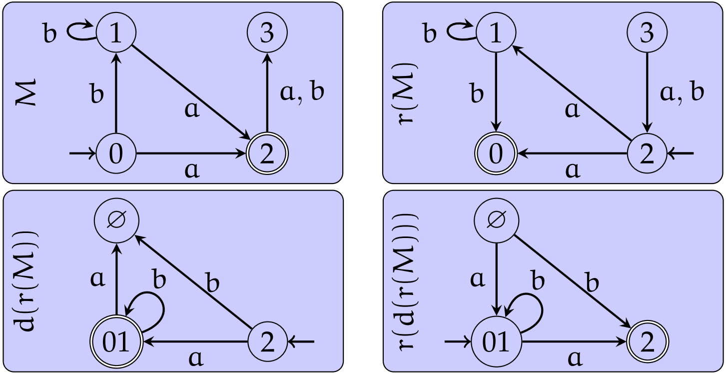

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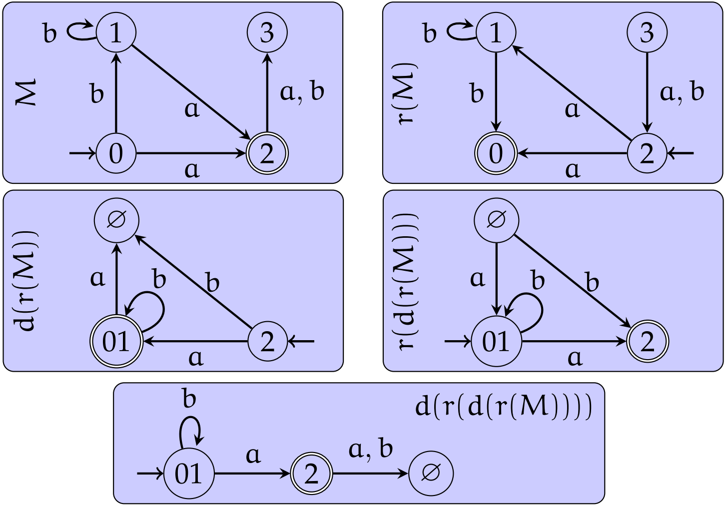

Minimization algorithms

final remarks

- There are many versions of the 'partitioning' algorithm. General idea is to form equivalence classes based on *right-language* of each state.
- Partitionin algorithm has $O(n \log n)$ complexity
- 'Double reversal' algorithm has exponential worst-time complexity
- Double reversal algorithm can also be used with NFA's (resulting in the minimal equivalent DFA – NFA minimization is intractable)
- In practice, there is no clear winner, different algorithms run faster on different input

Regular expressions

- Another way to specify a regular language (RL) is use of *regular expressions* (RE)
- Every RL can be expressed by a RE, and every RE defines a RL
- A RE x defines a RL $\mathcal{L}(\mathtt{x})$
- Relations between RE and RL

  - $\mathcal{L}(\emptyset) = \emptyset$,
  - $\mathcal{L}(\epsilon) = \epsilon$,
  - $\mathcal{L}(\mathtt{a}) = a$
  - $\mathcal{L}(\mathtt{ab}) = \mathcal{L}(a)\mathcal{L}(b)$
  - $\mathcal{L}(\mathtt{a*}) = \mathcal{L}(a)^*$
  - $\mathcal{L}(\mathtt{a|b}) = \mathcal{L}(a) \cup \mathcal{L}(b)$ (some author use the notation `a+b`, we will use `a|b` as in many practical implementations)

  where, $a, b \in \Sigma$, ϵ is empty string, $\emptyset$ is the language that accepts nothing (e.g., $\Sigma^* - \Sigma^*$)
- Note: no stadard complement operation in RE

Regular

some extensions

- Concatenation (`ab`), Kleene star (`a*`) and union (`a|b`) are the common operations
- Parentheses can be used to group the sub-expressions. Otherwise, the priority of the operators as specified above `a|bc*` = `a|(b(c*))`
- In practice some short-hand notations are common
  - `.` = `(a1|...|an)`, for $\Sigma = \{a_1, \ldots, a_n\}$
  - `a+` = `aa*`
  - `[a-c]` = `(a|b|c)`
  - `[^a-c]` = `. - (a|b|c)`
  - `\d` = `(0|1|...|8|9)`
  - …
- And some non-regular extensions, like `(a*)b\1` (sometimes the term *regexp* is used for expressions with non-regular extensions)

Some properties of regular expressions

Kleene algebra

These identities are often used to simplify regular expressions.

- $\epsilon \texttt{u} = \texttt{u}$
- $\varnothing \texttt{u} = \varnothing$
- $\texttt{u(vw)} = \texttt{(uv)w}$
- $\varnothing \texttt{*} = \epsilon$
- $\epsilon \texttt{*} = \epsilon$
- $\texttt{(u*)*} = \texttt{u*}$
- $\texttt{u|v} = \texttt{v|u}$
- $\texttt{u|u} = \texttt{u}$
- $\texttt{u|}\varnothing = \texttt{u}$
- $\texttt{u|}\epsilon = \texttt{u}$
- $\texttt{u|(v|w)} = \texttt{(u|v)|w}$
- $\texttt{u(v|w)} = \texttt{uv|uw}$
- $\texttt{(u|v)*} = \texttt{(u*|v*)*}$

Note: most of these follow from set theory, and some can be derived from others.

Some properties of regular expressions

Kleene algebra

These identities are often used to simplify regular expressions.

- ϵu = u
- $\varnothing$u = $\varnothing$
- u(vw) = (uv)w
- $\varnothing$* = ϵ
- ϵ* = ϵ
- (u*)* = u*
- u|v = v|u
- u|u = u
- u|$\varnothing$ = u
- u|ϵ = u
- u|(v|w) = (u|v)|w
- u(v|w) = uv|uw
- (u|v)* = (u*|v*)*

An exercise

Simplify a|ab*

Note: most of these follow from set theory, and some can be derived from others.

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- $\varnothing * = \epsilon$
- $\epsilon * = \epsilon$
- $\mathtt{(u*)*} = \mathtt{u*}$
- $\mathtt{u|v} = \mathtt{v|u}$
- $\mathtt{u|u} = \mathtt{u}$
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An exercise

Simplify $\mathtt{a|ab*}$

$$\mathtt{a|ab*} \quad = \quad \mathtt{a}\epsilon\mathtt{|ab*}$$

Note: most of these follow from set theory, and some can be derived from others.

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Simplify $\texttt{a|ab*}$

$$\begin{aligned} \texttt{a|ab*} &= \texttt{a}\epsilon\texttt{|ab*} \\ &= \texttt{a(}\epsilon\texttt{|b*)} \end{aligned}$$

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An exercise

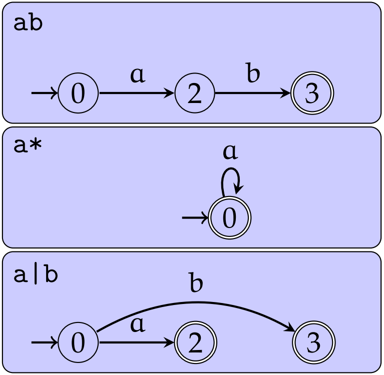
Simplify a|ab*

$$\begin{aligned} \mathtt{a|ab*} &= \mathtt{a}\epsilon\mathtt{|ab*} \\ &= \mathtt{a}(\epsilon\mathtt{|b*}) \\ &= \mathtt{ab*} \end{aligned}$$

Note: most of these follow from set theory, and some can be derived from others.

Converting between RE and FSA

Converting to NFA is easy:



Note the similarity with operations on regular languages discussed earlier.

- For more complex expressions, one can replace the paths for individual symbols with corresponding automata
- Using ϵ transitions may be ease the task
- The reverse conversion (from automata to regular expressions) is also easy:
  - identify the patterns on the left, collapse paths to single transitions with regular expressions

Wrapping up

- FSA and regular expressions express regular languages
- FSA have two flavors: DFA, NFA (or maybe three: ϵ-NFA)
- DFA recognition is linear
- Any NFA can be converted to a DFA (in worst case number of nodes increase exponentially)
- Regular languages and FSA are closed under
  - Concatenation
  - Kleene star
  - Complement
  - Reversal
  - Union
  - Intersection
- Every FSA has a unique minimal DFA

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Next:

- Finite state transducers (FSTs)
- Applications of FSA and FSTs

References / additional reading material

- Hopcroft and Ullman (1979, Ch. 2&3) (and its successive editions) covers (almost) all topics discussed here
- Jurafsky and Martin (2009, Ch. 2)
- Other textbook references include:
  - Sipser (2006)
  - Kozen (2013)

References / additional reading material (cont.)

Hopcroft, John E. and Jeffrey D. Ullman (1979). *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley. ISBN: 9780201029888.

Jurafsky, Daniel and James H. Martin (2009). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition*. second. Pearson Prentice Hall. ISBN: 978-0-13-504196-3.

Kozen, Dexter C. (2013). *Automata and Computability*. Undergraduate Texts in Computer Science. Berlin Heidelberg: Springer.

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